

Complex-space singularities of 2D Euler flow in Lagrangian coordinatesTakeshi Matsumoto,^{1,2,*} Jérémie Bec,² and Uriel Frisch²¹*Dep. Physics, Kyoto Univ., Kitashirakawa Oiwakecho Sakyo-ku Kyoto, 606-8502, Japan*²*Labor. Cassiopée, UNSA, CNRS, OCA, BP 4229, 06304 Nice Cedex 4, France*

We show that, for two-dimensional space-periodic incompressible flow, the solution can be evaluated numerically in Lagrangian coordinates with the same accuracy achieved in standard Eulerian spectral methods. This allows the determination of complex-space Lagrangian singularities. Lagrangian singularities are found to be closer to the real domain than Eulerian singularities and seem to correspond to fluid particles which escape to (complex) infinity by the current time. Various mathematical conjectures regarding Eulerian/Lagrangian singularities are presented.

PACS numbers: 47.10.A-, 47.15.ki, 47.27.er

I. INTRODUCTION

Solutions to the incompressible Euler equation, starting from entire initial data (e.g. trigonometric polynomials), can be analytically continued to the complex space as long as they stay analytic in the real space. Furthermore it is known since the seventies that any singularities in the real space, if they exist, have to be preceded by complex-space singularities [1, 2]. In 2D Euler flows, it is known that initial real-space analyticity for periodic solutions is never lost. This was proven in Refs. [3, 4] in which it was shown that the distance $\delta(t)$ to the real domain of the nearest complex-space singularity, measured by the exponential falloff of the Fourier amplitude, decreases at large times at most as a double exponential. Actually, already twenty five years ago, spectral simulations with 256^2 Fourier modes indicated that the decrease is more like a simple exponential [5]. Spectral simulations at much higher resolutions, up to 8192^2 , which will be published elsewhere also indicate a behavior much closer to a single than to a double exponential.¹ The discrepancy between the simple and the double exponential is generally believed to be due to the phenomenon of *depletion*: the flow organizes itself into ribbon-like vortical structures in which the nonlinearity is almost completely suppressed (the nonlinearity would vanish identically if the flow depended on a single Cartesian coordinate). The same phenomenon also exists in three dimensions and could conceivably prevent finite-time blow up.

In three dimensions the Beale–Kato–Majda (BKM) theorem implies that any blow up must be accompanied by the unboundedness of the modulus of the vorticity in the real domain [7] (see also [8]). In two dimensions, when the initial vorticity is bounded, this is of course ruled out by vorticity conservation. More precisely, it is

ruled out in the real domain, but not in the complex domain. Actually, increasingly strong numerical evidence has been obtained indicating that the vorticity is infinite at complex singularities [9, 10, 11].² Such numerical results were obtained only for flows in which the initial stream function is a trigonometric polynomial (the 2D analogues of the famous Taylor–Green flow [12]), which are instances of entire functions, that is, analytic functions that have no singularity at finite complex locations.

The “experimental result” about infinite vorticity along the complex singularities in two dimensions has an important consequence: because the conservation of vorticity along fluid particle trajectories carries over to complex trajectories, (Eulerian) complex locations with infinite vorticity are associated with fluid particles initially at complex infinity; indeed, this is the only place where an entire function can be infinite. We were thus led to investigate the issue of (complex) singularities in Lagrangian coordinates. A Lagrangian singularity is a location at which the (analytic continuation of the) Lagrangian map goes singular. Could it be that for two-dimensional flow there are no (complex) Lagrangian singularities at finite distance? In other words: does the flow in Lagrangian coordinates preserve its initial entire character? A few years ago we performed very accurate numerical simulations, reported here for the first time, and we found strong evidence that the answer is “no”. W. Pauls and one of us (TM) [14] then found a very simple counterexample to the preservation of the entire character: the “AB flow” $\psi = \sin x_1 \cos x_2$ is an entire steady solution to the 2D Euler equation in Eulerian coordinates. For this flow, the trajectories of fluid particles can be expressed by elliptic functions and it was shown that, for any real positive time t , there exist complex initial locations of fluid particles which are mapped to infinity at time t and which thus are Lagrangian singularities.

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¹ We nevertheless conjecture that by suitable analytic regularization of the flow considered in Ref. [6] long-lasting transients with a double exponential decrease can be achieved.

² We have tried but failed to derive such a result from a complex version of the the BKM argument.

There is a considerable renewal of interest in the Lagrangian structure of flows, both from a theoretical and experimental point of view (such issues frequently came up during the Euler conference). It is thus of interest to show that the Lagrangian description of flows can be obtained numerically with an accuracy comparable to that available by spectral methods for the Eulerian description. The present paper is organized as follows. In Section II we describe two numerical algorithms, which can be used for Lagrangian integration. In Section III we apply this to the identification of complex Lagrangian singularities. Here, all numerical studies are presented for the (unsteady) 2D flow with the simple initial condition

$$\psi_0 = \cos x_1 + \cos 2x_2, \quad (1)$$

which has been used in Refs. [9, 10, 11]; key results are also checked with the flow

$$\psi_0 = \cos x_1 + \cos 2x_2 + \sin(2x_1 - x_2), \quad (2)$$

which has less symmetry than (1). Some concluding remarks, with emphasis on mathematical conjectures, are presented in Section IV.

II. NUMERICAL SOLUTION IN LAGRANGIAN COORDINATES WITH SPECTRAL ACCURACY

Our goal here is to obtain the velocity field \mathbf{u} as a function of the Lagrangian location \mathbf{a} and the time t . This Lagrangian field will be denoted $\mathbf{u}_L(\mathbf{a}, t)$.

With simple boundary conditions, e.g. spatial periodicity, the easiest way to obtain high accuracy in a Eulerian simulation is to use a spectral or pseudo-spectral method [13]. For analytic flow, whose Fourier transform decreases exponentially at high wavenumbers, the truncation error will then also decrease exponentially with the resolution.

How does one carry this over to Lagrangian coordinates? In principle one can write an integro-differential equation for the (time-dependent) Lagrangian map $\mathbf{a} \mapsto \mathbf{x}$. This equation has however nonlinearities with denominators which are not easily handled numerically.

We present here two alternative methods, the spectral particle-tracking method (Section II A) and the spectral displacement-Newton method (Section II B).

A. Particle tracking method

Obviously, the Lagrangian velocity field can be obtained by composing the Eulerian velocity field $\mathbf{u}(\mathbf{x}, t)$ with the Lagrangian map $\mathbf{x}(\mathbf{a}, t)$. The former can be obtained by standard spectral integration. The latter is the solution of the characteristic equation

$$\partial_t \mathbf{x}(\mathbf{a}, t) = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t), \quad \mathbf{x}(\mathbf{a}, 0) = \mathbf{a}. \quad (3)$$

In the tracking method, we select a uniform grid of Lagrangian points and “track” the fluid particles by integrating (3) along all the relevant fluid particle trajectories. This can be done, e.g. using a fourth-order Runge-Kutta method. The problem is that, even if the initial positions coincide with Eulerian collocation points, this usually ceases to hold subsequently. Hence the Eulerian field must be interpolated. In order not to lose the spectral accuracy, the interpolation can be done using the Fourier series representation

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (4)$$

A difficulty is that, since the relevant \mathbf{x} ’s are not collocation points, the velocities given by (4) cannot be evaluated using fast Fourier transforms but must be calculated “naïvely” in $O(N^4)$ operations if we use an $N \times N$ grid. Furthermore this has to be done at every time step. Since the number of time steps needed to reach a given time t order unity is proportional to the resolution N , this method has a fairly large computational complexity $O(N^5)$ and thus also a significant accumulation of round-off errors. For large values of the resolution N (512 or more) the particle tracking method is not very practical unless we restrict the Lagrangian grid to being much coarser than the Eulerian grid.

B. Displacement-Newton method

This method makes use of the fact that the *inverse Lagrangian map* $\mathbf{a}(\mathbf{x}, t)$ satisfies, in Eulerian coordinates, the equation

$$\partial_t \mathbf{a} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \mathbf{a} = 0, \quad (5)$$

which just expresses the constancy of the Lagrangian location \mathbf{a} under advection by the velocity field. This equation can be solved along with the basic Euler equation, both in Eulerian coordinates. This will however yield a map which still has to be inverted to obtain the direct Lagrangian map.

For periodic boundary conditions the direct and inverse maps are not periodic and it is more convenient to work with the displacement field, here defined as

$$\mathbf{d}(\mathbf{x}, t) \equiv \mathbf{a}(\mathbf{x}, t) - \mathbf{x}. \quad (6)$$

It follows from (5) and (6) that the displacement satisfies the following equation in the Eulerian coordinates

$$\partial_t \mathbf{d}(\mathbf{x}, t) + (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{d}(\mathbf{x}, t) = -\mathbf{u}(\mathbf{x}, t), \quad (7)$$

with the initial condition $\mathbf{d}(\mathbf{x}, 0) = \mathbf{0}$. This equation can be solved along with the Euler equation to obtain the displacement in Eulerian coordinates on a uniform grid of $N \times N$ collocation points.

Then comes the difficult step, namely the inversion. For this we define the off-grid displacement, as above, by

its Fourier series, extended off-grid and we try to find the \mathbf{x} locations associated to a set of Lagrangian collocation points on the regular grid $\mathbf{A} = (2\pi i/N, 2\pi j/N)$, $i, j = 0, \dots, N-1$. We then determine the direct Lagrangian map $\mathbf{x}(\mathbf{A}, t)$ as the solution of the equation

$$\mathbf{d}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}, t) - \mathbf{x}. \quad (8)$$

First we determine an approximate on-grid solution $\mathbf{X}(\mathbf{A}, t)$ by finding from the inverse map the \mathbf{a} point nearest to \mathbf{A} and its inverse Lagrangian antecedent \mathbf{X} . We then set $\mathbf{x} = \mathbf{X} + \delta\mathbf{x}$ and refine the solution of (8) by using a standard Newton method. This requires the calculation of off-grid values of derivatives, which are again obtained from “naïve” evaluations of the corresponding Fourier series

$$\frac{\partial \mathbf{d}}{\partial x_j} = \sum_{\mathbf{k}} i k_j \hat{\mathbf{d}}(\mathbf{k}, t) e^{i \mathbf{k} \cdot \mathbf{x}}, \quad (9)$$

where $\hat{\mathbf{d}}(\mathbf{k}, t)$ are the Fourier coefficients of the displacement (evaluated in Eulerian coordinates). For each stage of the Newton iteration $O(N^4)$ operations are required. The number of stages needed to achieve an accuracy ϵ consistent with double precision is typically five. If the number of output times at which we want to evaluate the Lagrangian velocity field is much smaller than the resolution N , the displacement-Newton method is much faster than particle tracking.

III. RESULTS

We have applied the two methods described in the previous section to the flow with the initial condition (1). The methods give consistent results but the highest resolution (here $N = 512$) is more easily achieved with the displacement-Newton method, which has been used to obtain the results reported here.

The solution of the Euler equation

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p, \quad \nabla \cdot \mathbf{u} = 0, \quad (10)$$

together with the displacement equation (7) was obtained by a standard pseudo-spectral method with two-thirds dealiasing and a fourth-order Runge–Kutta temporal integration.

Then we applied the displacement-Newton method (with five iterations) and $\epsilon = 10^{-14}$. The results were checked by computing the Lagrangian vorticity, which should be equal to its initial value for 2D Euler flow, and was indeed found to be so with an accuracy of 10^{-10} .

In order to locate complex-space singularities for the Lagrangian solution, we applied the tracing method [5]: the Lagrangian solution is represented by its Fourier series

$$\mathbf{u}_L(\mathbf{a}, t) = \sum_{\mathbf{k}} \hat{\mathbf{u}}_L(\mathbf{k}, t) e^{i \mathbf{k} \cdot \mathbf{a}}. \quad (11)$$

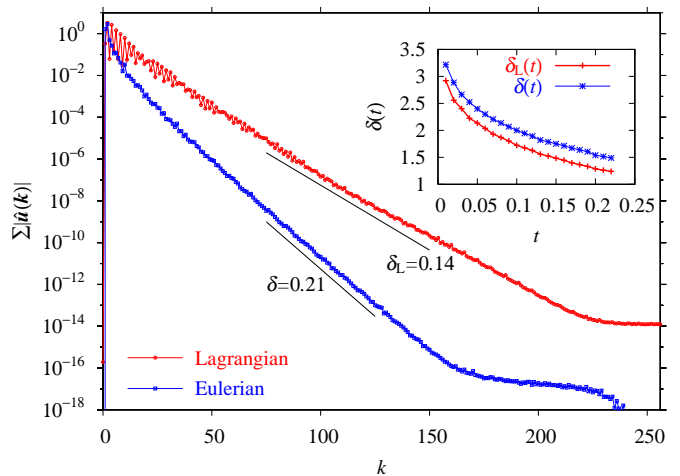


FIG. 1: Shell-summed amplitudes of Eulerian and Lagrangian velocities at time $t = 1.245$ in lin-log coordinates. The initial velocity is given by (1). Inset: time variation (at short times) of the width of the analyticity strip in Eulerian coordinates ($\delta(t)$) and Lagrangian coordinates ($\delta_L(t)$).

Then the following asymptotic representation is used for the shell-summed high wavenumber Fourier amplitudes ([10])

$$\sum_{k \leq |\mathbf{k}| < k+1} |\hat{\mathbf{u}}_L(\mathbf{k}, t)| \simeq C(t) k^{\alpha(t)} \exp[-\delta_L(t)k]. \quad (12)$$

Here $\delta_L(t)$ is the width of the Lagrangian analyticity strip, that is the distance at time t from the real domain of the nearest (Lagrangian) complex-space singularity. The same analysis is applied also to the Eulerian velocity.

Figure 1 shows the wavenumber dependence of the shell-summed amplitudes for the Eulerian and Lagrangian velocities at the same time, chosen in such a way that there is only a modest range of wavenumbers at high $k \equiv |\mathbf{k}|$ at which the rounding errors swamp the (roughly) exponential signal. They both exhibit exponential decay from which the Eulerian $\delta(t)$ and its Lagrangian counterpart $\delta_L(t)$ are measured. It is seen that the Lagrangian δ is significantly smaller than the Eulerian one. Actually, $\delta_L(t) < \delta(t)$ holds for all times $t < 1.245$ (the latest time analyzed). We also checked that the inequality $\delta_L < \delta$ holds for the flow with the initial condition (2) which has less symmetry than (1).

The exponential decay with the wavenumber k of the shell-summed Lagrangian Fourier amplitude is strong evidence that there are singularities of the Lagrangian velocity $\mathbf{u}_L(\mathbf{a}, t)$ at a finite distance from the real domain; thus it cannot be an entire function. We also obtained numerical evidence that the Lagrangian map has the same locations of complex Lagrangian singularities as the Lagrangian velocity and that the inverse Lagrangian map has the same locations of complex Eulerian singularities as the Eulerian velocity. For the very simple Eulerian steady flow investigated in Ref. [14], Lagrangian singu-

larities are mapped to Eulerian (complex) infinity. Is this also the case for the present flow which has non-trivial Eulerian dynamics? Here, the answer appears to be “yes”. Specifically, let \mathbf{a}_* be a Lagrangian singular location corresponding to time $t > 0$, say the one closest to the real domain or one near this position. Does $\mathbf{d}(\mathbf{a}, t) \equiv \mathbf{a} - \mathbf{x}(\mathbf{a}, t)$ tend to infinity as $\mathbf{a} \rightarrow \mathbf{a}_*$? In principle we can find the scaling law of any component of \mathbf{d} as $\mathbf{a} \rightarrow \mathbf{a}_*$, if we have sufficiently accurate high-resolution data for the Fourier transform of $\mathbf{d}(\mathbf{a})$ at high wavenumbers. This is explained in Section 4.2 of Ref. [11]. It requires the determination not only of the exponential decrement δ_L but of the exponent of the algebraic prefactor in front of the exponential which controls the nature of the singularity in complex \mathbf{a} -space. With a resolution of only 512^2 , such exponents are rather poorly determined. It is likely that both components of $\mathbf{d}(\mathbf{a})$ blow up as $s^{-\beta}$ where s is the modulus of $\mathbf{a} - \mathbf{a}_*$ and the exponent β is about $3/2$ but with an error bar so large that a negative value cannot be completely ruled out.³ We shall revisit such issues from a theoretical point of view in the concluding section.

IV. CONCLUDING REMARKS

We have shown that the simple 2D incompressible non-steady flow with the initial condition (1) has complex singularities not only in Eulerian but also in Lagrangian coordinates. The Lagrangian singularities are significantly closer to the real domain than the Eulerian ones. A possible interpretation of this was given by S. Orszag (private communication 2003): in Eulerian coordinates the build up of singularities is slowed down by the aforementioned phenomenon of depletion, whereas in Lagrangian coordinates any flow which is non-uniform will keep changing non-trivially, even if it is steady in Eulerian coordinates. To illustrate this we have shown in Figure 2 the (Eulerian) Laplacian of the vorticity $\nabla^2\omega$ in both Eulerian and Lagrangian coordinates. The former representation displays strongly depleted ribbon-shaped structures, not seen in the latter.

Now we wish to comment on the results concerning the analytic structure in Lagrangian coordinates and on possible generalizations to other 2D flow with space-periodic entire initial data. The most obvious result is that, since the vorticity remains unchanged along fluid particle trajectories in 2D, the Lagrangian vorticity field stays entire for all times and thus devoid of any singularities other than at complex infinity. The Lagrangian velocity field

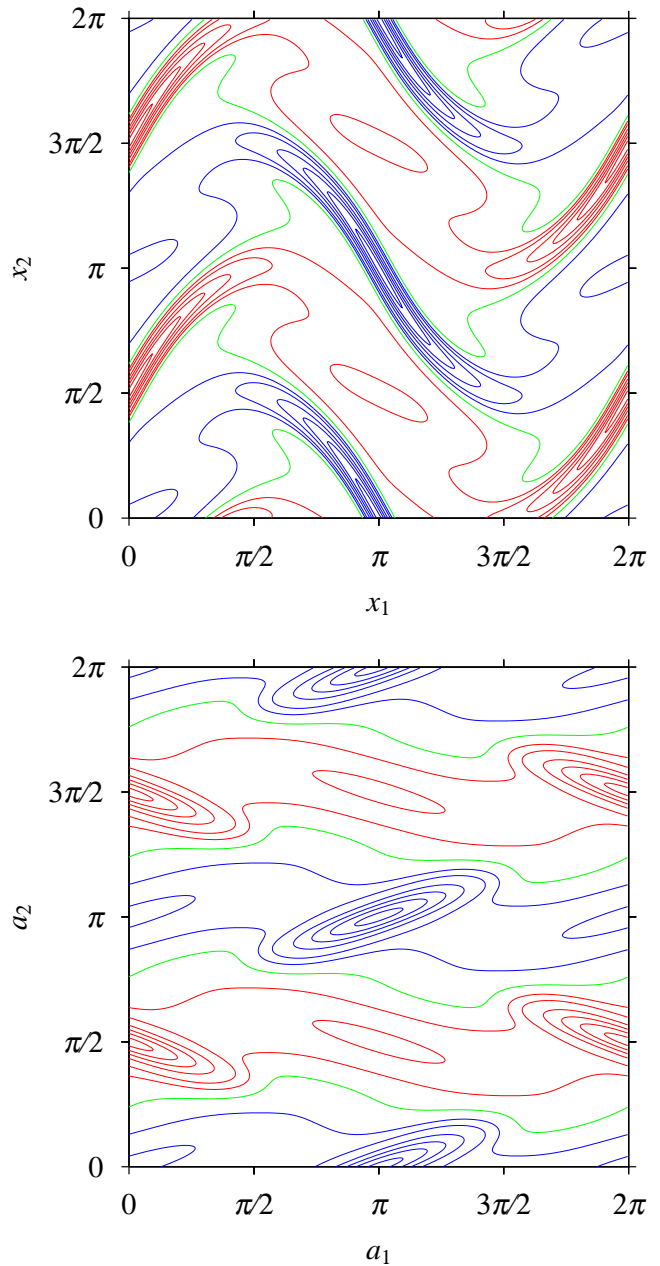


FIG. 2: Contours of the Laplacian of the vorticity shown for the same flow as in Figure 1, shown at $t = 1.245$. Upper figure: Eulerian coordinates; lower figure: Lagrangian coordinates. For both figures the contour values are 0, ± 12.5 , ± 25 , ± 50 , ± 75 , ± 100 , ± 125 ; red: positive, green: zero, blue: negative.

and the Lagrangian map both have complex singularities (presumably along one-dimensional complex manifolds) and the numerical evidence is that these are at the same locations. Proving this partially can perhaps be done by writing the velocity in terms of the vorticity using the (periodicity-modified) Biot–Savart integral representation and then making the change of variable from Eulerian to Lagrangian coordinates. On the resulting integral, using the fact that the initial vorticity is entire,

³ We have applied the same method of analysis to the behavior of $\mathbf{d}(\mathbf{x})$ when approaching a Eulerian singularity at \mathbf{x}_* . The displacement seems again to diverge with an exponent β around $3/2$ (implying also the divergence of the Eulerian vorticity) but the quality of the scaling is again dubious.

it may be possible to show that if the Lagrangian map $\mathbf{x}(\mathbf{a}, t)$ is analytic for some (complex) \mathbf{a} , the same holds for the Lagrangian velocity.

One of the most striking results reported in Section III, but one for which the evidence is a bit shaky, is that Lagrangian singularities at time $t > 0$ correspond to fluid particles which at time t escape to infinity. Here are some observations which could be useful in proving this. The idea is to show that there is a contradiction if at time $t > 0$ a Lagrangian singularity \mathbf{a}_* at a finite location is mapped to a point \mathbf{x}_* which is not at infinity. Indeed, if \mathbf{x}_* is at finite distance, from the fact that the Jacobian of the Lagrangian map is one, it follows that \mathbf{x}_* must be a singularity of the inverse Lagrangian map $\mathbf{x} \mapsto \mathbf{a}$. The Eulerian vorticity can be obtained by composing the inverse Lagrangian map and the initial (entire) vorticity. Composing a function singular at \mathbf{x}_* with one which is entire does not necessarily yield a singular function. Perhaps with some extra work it can be proved that the Eulerian vorticity is indeed singular at \mathbf{x}_* . We already pointed out in the Introduction that for 2D space-periodic initially entire flow there is numerical evidence that the vorticity is infinite at (complex) Eulerian singularities. If this can also be proved, it then follows that \mathbf{a}_* is at infinity and thus we have a contradiction.

The global picture emerging from all this is (tentatively) the following: for entire periodic initial data in 2D, the solutions of the incompressible Euler equation have complex Eulerian singularities corresponding to fluid par-

ticles initially at infinity and Lagrangian singularities corresponding to fluid particles currently at infinity. In both coordinates singularities correspond to some particle escaping to infinity; this mechanism for incompressible fluids is very different from the one operating for the one or multi-dimensional *compressible* Burgers equation for which singularities are mostly associated to the vanishing of the Jacobian of the the Lagrangian map (see, e.g., Ref.[15]).

We cannot at present rule out that the same scenario holds in three dimensions but it may not be consistent with real blow up. Of course, there are major differences in 3D; for example, vorticity is not conserved. However, the Lagrangian numerical techniques presented in this paper are easily extended to the three-dimensional case.

Acknowledgments

We are very grateful to W. Pauls for many useful remarks. Thanks are also due to C. Bardos and S. Orszag. TM was supported by the Grant-in-Aid for the 21st Century COE “Center for Diversity and Universality in Physics” from the Japanese Ministry of Education, by the Japanese Ministry of Education Grant-in-Aid for Young Scientists [(B), 15740237, 2003] and by the French Ministry of Education.

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